

Solvable Examples of Drift and Diffusion of Ions in Non-uniform Electric Fields

R. N. Cahn

*Lawrence Berkeley National Laboratory
Berkeley, CA, 94720, USA
E-mail: rncahn@lbl.gov*

J. D. Jackson

*Lawrence Berkeley National Laboratory
Berkeley, CA, 94720, USA
E-mail: jdjackson@lbl.gov*

ABSTRACT: The drift and diffusion of a cloud of ions in a fluid are distorted by an inhomogeneous electric field. If the electric field carries the center of the distribution in a straight line and the field configuration is suitably symmetric, the distortion can be calculated analytically. We examine the specific examples of fields with cylindrical and spherical symmetry in detail assuming the ion distributions to be of a generally Gaussian form. The effects of differing diffusion coefficients in the transverse and longitudinal directions are included.

KEYWORDS: Gaseous detectors, Detector modelling and simulations II (electric fields, charge transport, multiplication and induction, pulse formation, electron emission, etc), time projection chambers, Particle tracking detectors.

Contents

1. Introduction	1
2. Preliminaries and initial Ansatz	2
3. Restriction to rectilinear motion and symmetric geometry	3
4. Example of cylindrical geometry	4
5. Inward drift	9
6. Different coefficients of diffusion for longitudinal and transverse directions	9
7. Diffusion of a track segment	10
A. Solution of Ricatti equations for cylindrical geometry	12
B. Example of spherical geometry	13

1. Introduction

We address the time and spatial evolution of the ionization produced along a charged particle's track in a fluid (gas or liquid) subjected to a non-uniform, time-independent electric field. The prototypical calculation is for a localized ion density produced at some point \mathbf{R}_0 at time $t = 0$, described by a delta function $\delta(\mathbf{r} - \mathbf{R}_0)$. Of interest is the diffusion in both the direction of the electric field and transverse to it, which governs the distributions in arrival time and transverse position at some detecting surface. We focus on a volume that is bounded by cylindrical electrodes that produce a radial electric field, but begin with more general considerations. Spherical geometry is considered in an appendix.

The motion of electrons and atomic or molecular ions in electromagnetic fields has a huge experimental and theoretical literature. See, for example, the texts and cited literature of Huxley and Crompton[1] and Mason and McDaniel[2]. An extensive literature about diffusion of charged particles in plasmas also exists, but conditions there are far from our concern of relatively few ions moving in a neutral medium.

We make the simplifying assumption that the ion mobility μ is independent of electric field strength for the range of E/N of interest ($(E/N)_{max} \approx 30$ Td[3]; see Figs.7-3-2 to 7-3-6, in Ref. [2]). We recognize the presence of different diffusion constants D_L and D_T in the longitudinal (field) and transverse directions, respectively, at higher field strengths.[4, 2] It appears, however, that typically

$D_L/D_T \approx 1$ for $E/N(\text{Td}) < 10$ and $D_L/D_T < O(1.5)$ for larger E/N . See, for example, figure 5-2-4 of Ref. [2]. We hypothesize a regime in which the microscopic (atomic) processes that establish the local drift velocity and diffusion parameters occur on length and time scales small compared to the changes in magnitude or direction of the electric field or the average drift position. Such a regime permits the use of the simple diffusion equation rather than the full Boltzmann equation.

2. Preliminaries and initial Ansatz

Ions in a fluid and subject to an electric field can be described by a charge density $\psi(\mathbf{r}, t)$ satisfying the diffusion equation

$$\frac{d\psi}{dt} = \frac{\partial\psi}{\partial t} + \mathbf{v} \cdot \nabla\psi = D\nabla^2\psi. \quad (2.1)$$

We take the velocity \mathbf{v} to be given by the mobility μ times the electric field \mathbf{E} . We imagine that from the prescribed electric field and mobility we can calculate the classical trajectory $\mathbf{r}_0(t)$ of a point charge located at $\mathbf{r} = \mathbf{R}_0$ at time $t = 0$. We indicate the velocity along the path by $\mathbf{v}_0(t)$ and, in particular,

$$\mathbf{v}_0(t) = \mu\mathbf{E}(\mathbf{r}_0(t)). \quad (2.2)$$

The trajectory is given implicitly as

$$\mathbf{r}_0(t) = \mathbf{R}_0 + \int_0^t dt' \mu\mathbf{E}(\mathbf{r}_0(t')). \quad (2.3)$$

We are seeking a solution proportional to $\delta(\mathbf{r} - \mathbf{R}_0)$ at $t = 0$. We begin with an Ansatz for ψ :

$$\psi = \frac{[\det(I+g)]^{1/2}}{(4\pi Dt)^{3/2}} \exp\left\{-[\mathbf{r} - \mathbf{r}_0(t)]_i [\delta_{ij} + g(t)_{ij}] [\mathbf{r} - \mathbf{r}_0(t)]_j / (4Dt)\right\}. \quad (2.4)$$

The factor $[\det(I+g)]^{1/2}$ ensures that ψ is normalized properly:

$$\int d^3\mathbf{r} \psi = 1, \quad (2.5)$$

which must be the case if we solve the initial equation, which derives from the continuity equation. In the absence of the electric field, the familiar solution is given by Eq.(2.4) with $g_{ij} = 0$ and $\mathbf{r}_0(t) = \mathbf{r}_0$. The basic assumption is that the form is Gaussian with respect to the difference between the point at which the density is evaluated and the location of the classical trajectory at that time, $\mathbf{r}_0(t)$. The form is motivated by the observation that, not only in the absence of the electric field, but also for a constant electric field, the solution is given by Eq.(2.4) with $g_{ij} = 0$. We calculate directly the following terms: We calculate the time and spatial derivatives writing $N(t)$ for $[\det(I+g)]^{1/2}$ and ρ_i for $(\mathbf{r} - \mathbf{r}_0(t))_i$:

$$\begin{aligned} 4Dt \frac{\partial\psi}{\partial t} &= \rho_i (\delta_{ij} + g_{ij}) \rho_j \frac{\psi}{t} - 6D\psi + 2\rho_i (\delta_{ij} + g_{ij}) v_{0j} \psi \\ &\quad - \rho_i \frac{dg_{ij}}{dt} \rho_j \psi + 4Dt \frac{d \ln N}{dt} \psi; \end{aligned} \quad (2.6)$$

$$4Dt\nabla_k\psi = -2(\delta_{jk} + g_{jk})\rho_j\psi; \quad (2.7)$$

$$\begin{aligned} 4D^2t\nabla_k\nabla_k\psi &= (\delta_{kj} + g_{kj})\rho_j(\delta_{kl} + g_{kl})\rho_l\frac{\psi}{t} - 2(\delta_{kk} + g_{kk})D\psi \\ &= \rho_j(\delta_{jl} + 2g_{jl} + g_{kj}g_{kl})\rho_l\frac{\psi}{t} - 2(3 + g_{kk})D\psi; \end{aligned} \quad (2.8)$$

$$4Dt\mu\mathbf{E}\cdot\nabla\psi = -2\mu E_k(\delta_{kj} + g_{jk})\rho_j\psi. \quad (2.9)$$

We now expand $\mathbf{E}(\mathbf{r})$ around $\mathbf{r}_0(t)$, keeping first order terms,

$$\mathbf{v}_0(t) - \mu\mathbf{E}(\mathbf{r}) = \mu(\mathbf{E}(\mathbf{r}_0(t)) - \mathbf{E}(r)) = -\mu\rho_i\partial_i\mathbf{E}(\mathbf{r}_0(t)) + \dots \quad (2.10)$$

The omitted terms in the expansion are of order $(\rho/L)^n$, $n \geq 1$ times the first order term, where L is the length scale over which the electric field changes appreciably. If the spread of the diffusion, $O(\rho_{max})$, is small compared to L , our approximation should be reliable.

We combine these into Eq.(2.1) to find the equations for g_{ij} :

$$\begin{aligned} \rho_i\frac{(\delta_{ij} + g_{ij})}{t}\rho_j - 6D + 4Dt\frac{d\ln N}{dt}\psi - \rho_i\frac{dg_{ij}}{dt}\rho_j \\ = \rho_j\frac{(\delta_{jl} + 2g_{jl} + g_{kj}g_{kl})}{t}\rho_l - 2(3 + g_{kk})D \\ + 2\mu\rho_i\partial_i E_k(\delta_{kj} + g_{jk})\rho_j. \end{aligned} \quad (2.11)$$

The terms with and without quadratic dependence on ρ must cancel separately. In fact, we find that the equations that derive from terms with quadratic dependence ensure that the terms without that dependence cancel, as well. The terms without the quadratic dependence give

$$1 - \frac{2t}{3}\frac{d\ln N}{dt} = 1 + \frac{1}{3}g_{kk} \quad (2.12)$$

The terms with quadratic dependence on ρ give

$$\frac{(\delta_{ij} + g_{ij})}{t} - \frac{dg_{ij}}{dt} = \frac{(\delta_{ji} + 2g_{ji} + g_{kj}g_{ki})}{t} + 2\mu\partial_i E_k(\delta_{kj} + g_{jk}) \quad (2.13)$$

or

$$\frac{dg_{ij}}{dt} + \frac{g_{ij} + g_{ik}g_{kj}}{t} + 2\mu\partial_i E_k(\delta_{kj} + g_{jk}) = 0 \quad (2.14)$$

3. Restriction to rectilinear motion and symmetric geometry

Only the symmetric part of g_{ij} enters ψ . Consistency requires that the antisymmetric part of Eq.(2.14) vanish:

$$(\delta_{ij} + g_{ij})\partial_k E_j = (\delta_{kj} + g_{kj})\partial_i E_j \quad (3.1)$$

The vanishing of the curl of \mathbf{E} is enough to take care of the δ_{ij} portion but we are left with the requirement

$$g_{ij}\partial_k E_j = g_{kj}\partial_i E_j \quad (3.2)$$

In general, the product of two symmetric matrices is not symmetric. We need to impose additional constraints. In particular, we shall suppose that the motion is rectilinear along the x direction, i.e. $\mathbf{r}(t) = r(t)\hat{\mathbf{x}}$, $\mathbf{E}(t) = E_x(t)\hat{\mathbf{x}}$. Write the electrostatic potential in the neighborhood of the line of motion as an expansion in y and z :

$$\begin{aligned}\Phi(x, y, z) = & \Phi(x, 0, 0) + y\Phi_y(x) + z\Phi_z(x) \\ & + \frac{1}{2}y^2\Phi_{yy}(x) + \frac{1}{2}z^2\Phi_{zz}(x) \\ & + xy\Phi_{xy}(x) + xz\Phi_{xz}(x) + yz\Phi_{yz}(x) + \dots\end{aligned}\tag{3.3}$$

By hypothesis the electric field along $y = 0, z = 0$ is entirely in the x direction. Then $\Phi_y(x) = \Phi_z(x) = \Phi_{xy}(x) = \Phi_{xz}(x) = 0$ and Eq.(3.3) becomes, to second order inclusive,

$$\Phi(x, y, z) = \Phi(x, 0, 0) + \frac{1}{2}y^2\Phi_{yy}(x) + \frac{1}{2}z^2\Phi_{zz}(x) + yz\Phi_{yz}(x)\tag{3.4}$$

If the surface charge distributions determining the electric field are symmetric under $y \rightarrow -y$ and $z \rightarrow -z$, the term $yz\Phi_{yz}$ vanishes along with $\Phi_y, \Phi_z, \Phi_{xy}$ and Φ_{xz} , and the only non-zero values of $\partial_i E_j$ along the line $(x, 0, 0)$ occur for $i = j$. Two clear examples are cylindrical symmetry with the z direction taken as the axis of the cylinder and spherical symmetry. In the former case all derivatives with respect to z vanish so $\partial_z E_z = 0$ and in the latter, the derivatives with respect to y and z are equal so $\partial_z E_z = \partial_y E_y$. See figure 1.

Under any of these circumstances we can write

$$\partial_i E_j = C_i(x)\delta_{ij}\tag{3.5}$$

and we can satisfy Eq.(3.2) with the ansatz

$$g_{ij} = g_i\delta_{ij}\tag{3.6}$$

i.e. both g_{ij} and $\partial_i E_j$ are diagonal along the linear trajectory path of Eq.(2.3). Now Eq.(2.14) simplifies to

$$\frac{dg_i}{dt} + \frac{g_i(1+g_i)}{t} + 2\mu\partial_i E_i(1+g_i) = 0\tag{3.7}$$

where there is no summation on i .

4. Example of cylindrical geometry

For a cylindrical geometry, uniform in the z direction and having azimuthal symmetry, with a radial electric field, we have

$$\mathbf{E} = \frac{A}{r}\hat{\mathbf{r}},\tag{4.1}$$

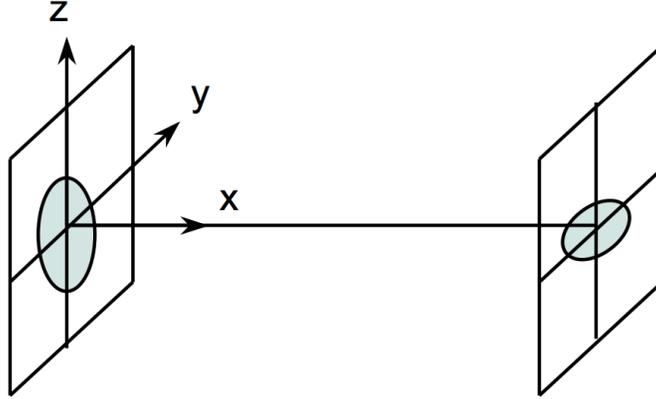


Figure 1. The Ansatz for ψ , Eq.(2.4), is applicable for rectilinear trajectories if the matrix $\partial_i E_j$ is diagonal, for a suitable choice of coordinates. Taking x along the direction of motion, a charge distribution (or potential) symmetric under $y \rightarrow -y$ and $z \rightarrow -z$ will ensure this. The figure shows such a trajectory and charge distribution. Here only the trajectory shown has the required symmetry. For the cylindrical and spherical geometries all radial trajectories have the symmetry required by our Ansatz.

where $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$ is now two-dimensional. We assume here that the signs of the charge and field are such that the force is radially outward.

We can anticipate the effect of the inhomogeneous electric field, Eq.(4.1). The portion of the cloud that is further out than the center of the cloud experiences a weaker field and thus moves more slowly than the outwardly moving center, while the portion of the cloud closer in to the central axis experiences a greater field and moves more rapidly than moving center. Thus the spread of the cloud in the radial direction should be smaller than in the absence of the electric field. On the other hand, the diverging electric field will tend to spread the cloud transversely, leading to a larger dispersion than in the absence of the electric field.

Take the initial position $\mathbf{R}_0 = (x_0, 0, 0)$. The motion is along the x axis where

$$\partial_x E_x = -\frac{A}{x^2}; \quad \partial_y E_y = \frac{A}{x^2}; \quad \partial_x E_y = 0 = \partial_y E_x, \quad (4.2)$$

satisfying the symmetry requirements for our Ansatz. Thus we have the diagonal form $g_{ij} = C_i(x)\delta_{ij}$ where

$$C_1(x) = -\frac{A}{x^2}, \quad (4.3)$$

$$C_2(x) = \frac{A}{x^2}, \quad (4.4)$$

$$C_3(x) = 0. \quad (4.5)$$

The trajectory is given by $\mathbf{r}_0(t) = r_0(t)\hat{\mathbf{x}}$, with

$$r_0(t)^2 = x_0^2 + 2\mu A t, \quad (4.6)$$

so that

$$2\mu \partial_x E_x = -\frac{2\mu A}{x_0^2 + 2\mu A t} = -2\mu \partial_y E_y. \quad (4.7)$$

Introduce

$$T = \frac{x_0^2}{2\mu A} \quad (4.8)$$

so that

$$r_0(t) = x_0 \sqrt{1 + \frac{t}{T}}. \quad (4.9)$$

In the characteristic time T the diffusion center would go a distance $x_0/2$ if the field had the constant value $E = A/x_0$. When $t = 3T$, the classical particle is twice as far from the axis of the cylinder in the $1/r$ field as it was at $t = 0$. From Eq.(3.7):

$$\begin{aligned} \frac{dg_1}{dt} &= -\frac{g_1(1+g_1)}{t} + \frac{1}{T+t}(1+g_1); \\ \frac{dg_2}{dt} &= -\frac{g_2(1+g_2)}{t} - \frac{1}{T+t}(1+g_2); \\ \frac{dg_3}{dt} &= -\frac{g_3(1+g_3)}{t}. \end{aligned} \quad (4.10)$$

Finally, writing $t = Ts$

$$\begin{aligned} \frac{dg_1}{ds} &= -\frac{g_1(1+g_1)}{s} + \frac{1}{1+s}(1+g_1); \\ \frac{dg_2}{ds} &= -\frac{g_2(1+g_2)}{s} - \frac{1}{1+s}(1+g_2); \\ \frac{dg_3}{ds} &= -\frac{g_3(1+g_3)}{s}. \end{aligned} \quad (4.11)$$

The equation for g_3 has the solution

$$g_3 = \frac{1}{Bs - 1} \quad (4.12)$$

where B is a constant. When $s = 0$, all the g_i are zero. It follows that $B = \infty$ and g_3 remains zero: Diffusion in the z direction is unchanged by the presence of the radial electric field.

Eqs.(4.11) for g_1 and g_2 are Riccati equations and can be solved analytically. The details of the solutions are given in Appendix A. The results are

$$g_1 = \frac{s}{2+s} \quad (4.13)$$

$$g_2 = -1 + \frac{s}{(1+s)\ln(1+s)} \quad (4.14)$$

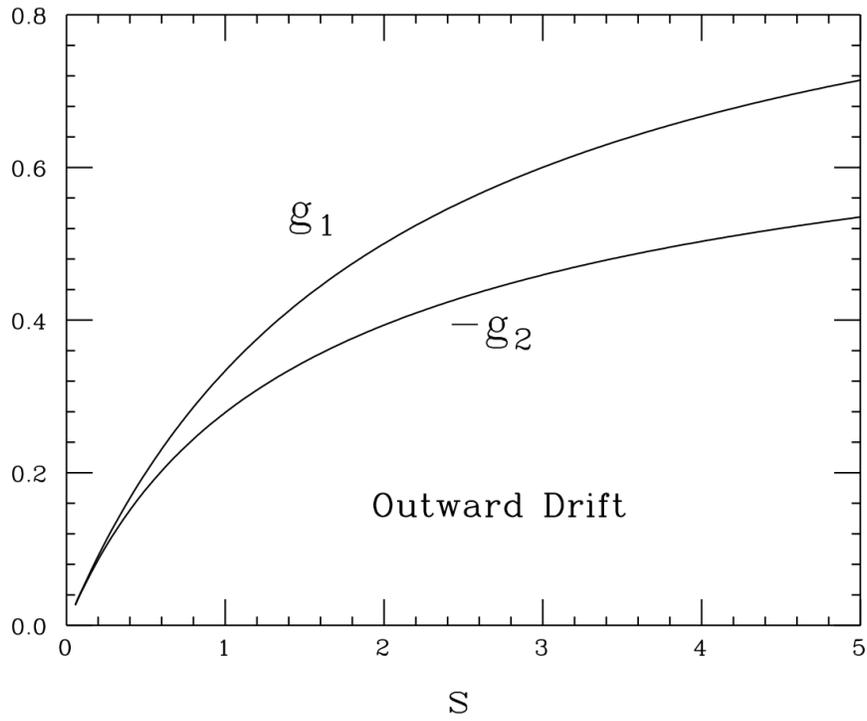


Figure 2. The functions appearing in the Gaussian description of the charge density ψ for outward drift, as a function of the dimensionless time variable $s = t/T$ for cylindrical geometry.

The behavior of the functions g_1 and g_2 is shown in figure 2.

The effective parallel and transverse spreads relative to $\sigma_0 = \sqrt{2Dt}$ are

$$\frac{\sigma_{\parallel}}{\sigma_0} = \sqrt{\frac{1}{1+g_1}}; \quad \frac{\sigma_{\perp}}{\sigma_0} = \sqrt{\frac{1}{1+g_2}}, \quad (4.15)$$

shown in figure 3. In the z -direction the spread is $\sigma_3 = \sigma_0$.

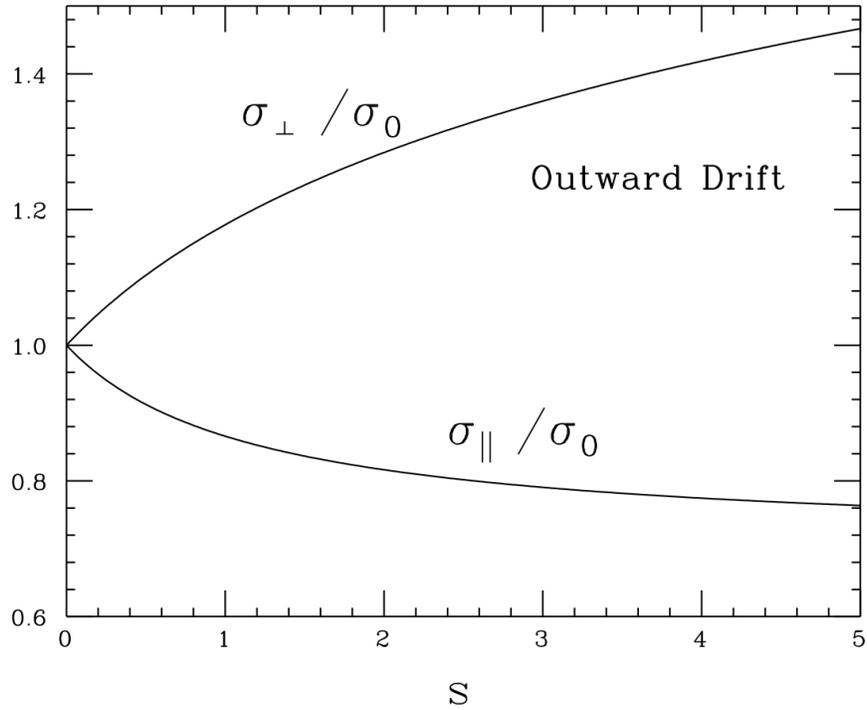


Figure 3. The effective dimensionless widths Eq.(4.15) for outward drift, in the directions parallel and perpendicular to the radial direction as functions of the dimensionless time variable $s = t/T$ for cylindrical geometry. If the diffusion constants differ in the longitudinal (parallel to the field) and transverse directions, multiply the ratio $\sigma_{\parallel}/\sigma_0$ by $\sqrt{D_L/D_T}$ and take $\sigma_0 = \sqrt{2D_T t}$. (see Sec. 6).

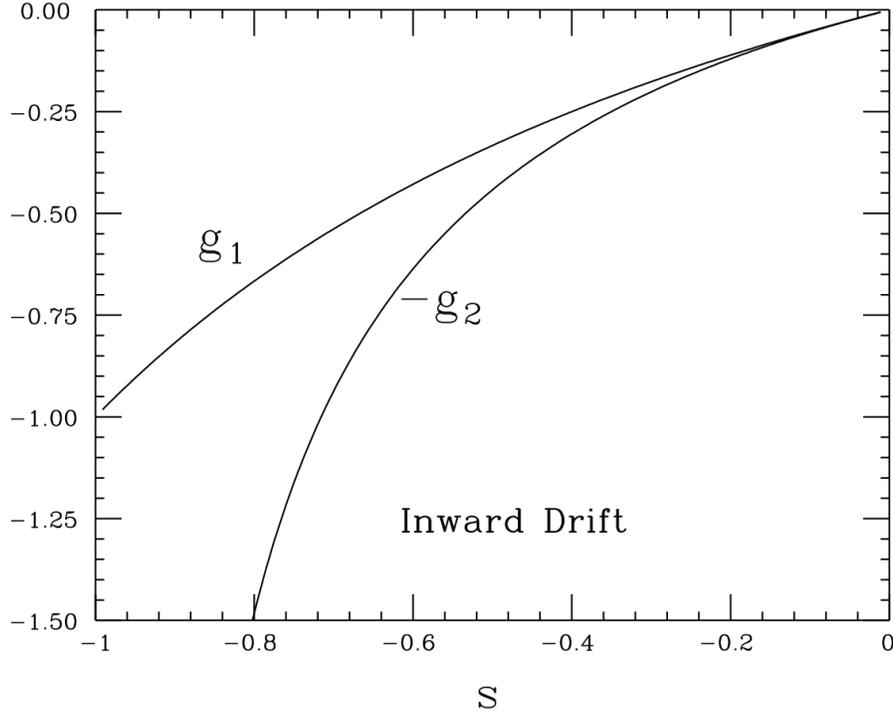


Figure 4. The functions appearing in the Gaussian description of ψ for inward drift in the cylinder. Negative values of s correspond to positive values of t according to $t = Ts$. For $s = -1$, $\mathbf{r}_0(t) = 0$.

5. Inward drift

We have imagined that the radial drift is outward, towards larger r . If the sign of the electric field is reversed, the derivation goes through much the same way except that the sign of T is negative. Since $t = Ts$, positive times then correspond to negative values of s . The center point of the diffusion cloud arrives at the central axis at $s = -1$. Because the field increases in the direction of motion, we expect the effect of the field to be opposite to what we found above: the dispersion should be greater in the radial direction and smaller in the transverse direction.

We show the results obtained for inward drift in Figs. 4 and 5.

6. Different coefficients of diffusion for longitudinal and transverse directions

It is straightforward to include the effect of having different diffusion coefficients D_L and D_T for the longitudinal and transverse directions. For the Ansatz we take

$$\psi = \frac{[\prod_j (1 + g_j)]^{1/2}}{(4\pi t)^{3/2} D_T D_L^{1/2}} \exp \left\{ - \sum_i \frac{\rho_i^2 (1 + g_i)}{4D_i t} \right\}. \quad (6.1)$$

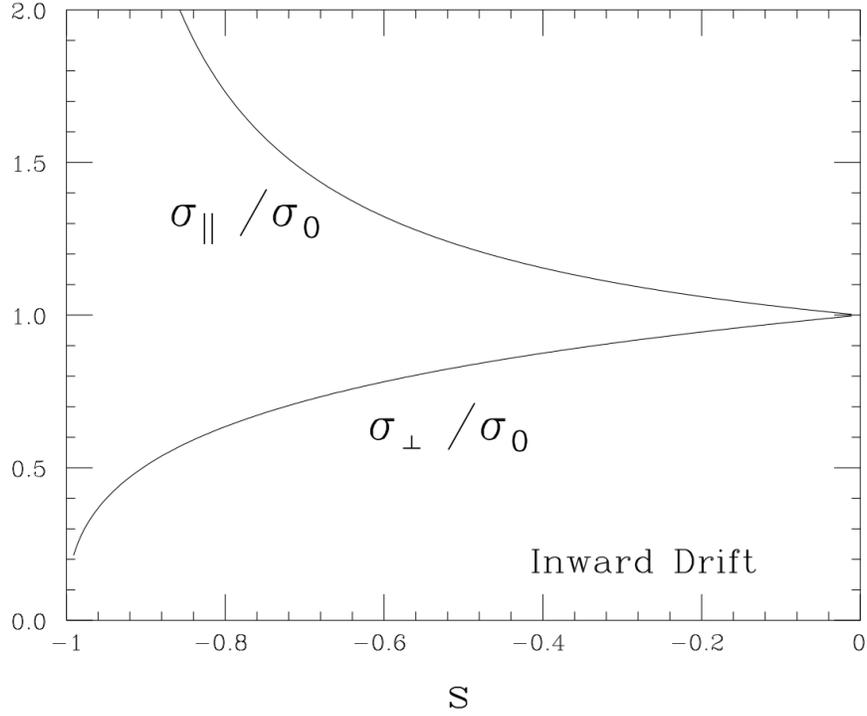


Figure 5. The effective dimensionless widths Eq.(4.15) in the directions parallel and perpendicular to the radial direction for inward drift in the cylinder. Negative values of s correspond to positive values of t according to $t = Ts$. If the diffusion constants differ in the longitudinal (parallel to the field) and transverse directions, multiply the ratio $\sigma_{\parallel}/\sigma_0$ by $\sqrt{D_L/D_T}$ and take $\sigma_0 = \sqrt{2D_T t}$ (see Sec. 6).

with $D_1 = D_L, D_2 = D_3 = D_T$. Exactly the same differential equations emerge, but setting $\sigma_0 = \sqrt{2D_T t}$ we have

$$\frac{\sigma_{\parallel}}{\sigma_0} = \sqrt{\frac{D_L}{D_T}} \frac{1}{\sqrt{1+g_1}} \quad (6.2)$$

$$\frac{\sigma_{\perp}}{\sigma_0} = \frac{1}{\sqrt{1+g_2}} \quad (6.3)$$

$$\frac{\sigma_3}{\sigma_0} = 1 \quad (6.4)$$

7. Diffusion of a track segment

To illustrate the possible use of our results, instead of a single point source arising at $\mathbf{r} = \mathbf{R}_0$ at $t = 0$, we consider the ionization along the path of a relativistic charged particle through a cylindrical drift chamber. The ionization trail is assumed linear (in both senses) and is, on the time scale relevant for the subsequent diffusion, deposited instantaneously. The resulting charge distribution is thus a

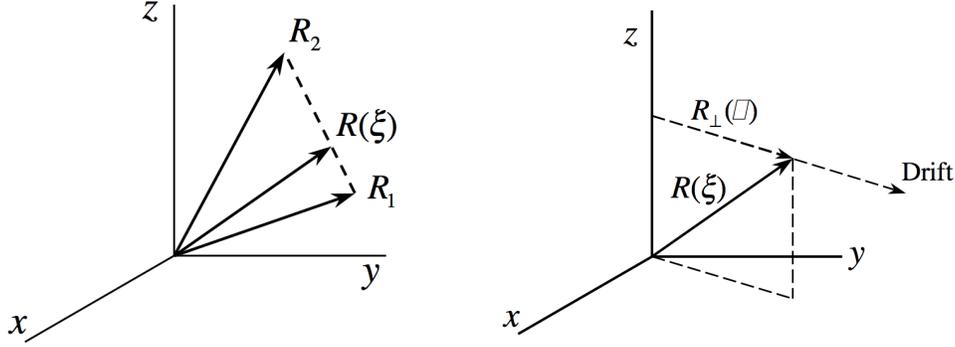


Figure 6. Left (a): Track segment ($\mathbf{R}_1, \mathbf{R}_2$) with typical starting point $\mathbf{R}(\xi)$. Right (b): Details of a typical starting point showing its decomposition $\mathbf{R}(\xi) = \mathbf{R}_\perp(\xi) + R_\parallel \hat{\mathbf{z}}$ into a piece along the drift direction and a piece parallel to the axis of the cylinder.

superposition of distributions of the form Eq.(6.1) for contributions along the particle's track. For definiteness we consider a track segment with end points $\mathbf{R}_1, \mathbf{R}_2$, as shown in figure (6a), with a typical point on the track labeled by $\mathbf{R}(\xi)$. For convenience, define the distance between the end points as $L = |\mathbf{R}_2 - \mathbf{R}_1|$. Then we have $\mathbf{R}(\xi) = \mathbf{R}_1 + (\xi/L)(\mathbf{R}_2 - \mathbf{R}_1)$. In figure (6b), the typical starting point $\mathbf{R}(\xi)$ is singled out. We define $R_\parallel(\xi) = \hat{\mathbf{z}} \cdot \mathbf{R}(\xi)$ and $\mathbf{R}_\perp(\xi) = \mathbf{R}(\xi) - \hat{\mathbf{z}}R_\parallel(\xi)$. Then the drift center's coordinate $\mathbf{r}_0(t)$ can be written according to Eq.(4.9) as

$$\mathbf{r}_0(t) = \mathbf{R}_\perp(\xi)[1 + t/T(\xi)]^{1/2} + \hat{\mathbf{z}}R_\parallel(\xi) \quad (7.1)$$

where from (30), since $x_0(\xi) = |\mathbf{R}_\perp(\xi)|$, we have

$$T(\xi) = \frac{|\mathbf{R}_\perp(\xi)|^2}{2\mu A} \quad (7.2)$$

In the exponent of ψ the Cartesian components of the coordinate difference, $(\mathbf{r} - \mathbf{r}_0(t))$, must be taken with respect to $\mathbf{R}_\perp(\xi)$ as the x -direction:

$$\rho_1 \equiv (\mathbf{r} - \mathbf{r}_0(t))_1 = (\mathbf{r} - \mathbf{r}_0(t)) \cdot \hat{\mathbf{R}}_\perp(\xi) \quad (7.3)$$

$$\rho_2 \equiv (\mathbf{r} - \mathbf{r}_0(t))_2 = (\mathbf{r} - \mathbf{r}_0(t)) \cdot (\hat{\mathbf{z}} \times \hat{\mathbf{R}}_\perp(\xi)) \quad (7.4)$$

$$\rho_3 \equiv (\mathbf{r} - \mathbf{r}_0(t))_3 = z - R_\parallel(\xi) \quad (7.5)$$

The normalized diffused charge density in space and time for the track segment can thus be written as a numerical integral over ψ given by Eq.(6.1),

$$\Psi(\mathbf{r}, t | \mathbf{R}_1, \mathbf{R}_2) = \frac{1}{L} \int_0^L d\xi \psi(\mathbf{r}, \mathbf{R}(\xi), t) \quad (7.6)$$

The functions $g_1(s)$ and $g_2(s)$ are to be evaluated at $s = t/T(\xi)$; $g_3(s) = 0$.

Acknowledgments: This work was supported by the Director, Office of Science, High Energy Physics, U.S. Department of Energy under Contract No. DE-AC02-05CH11231. We thank David Nygren for drawing our attention to this class of ionic diffusion problems.

A. Solution of Riccati equations for cylindrical geometry

The procedure for solution of the Riccati equations can be found in Refs.[6, 7]. The general form is

$$y' = q_0(x) + q_1(x)y + q_2(x)y^2. \quad (\text{A.1})$$

To find the solution we consider the associated second order homogeneous linear differential equation

$$u'' - P(x)u' + Q(x)u = 0 \quad (\text{A.2})$$

with

$$Q = q_2q_0; \quad P = q_1 + \frac{q_2'}{q_2}. \quad (\text{A.3})$$

Then the solution to the original equation is

$$y = -\frac{u'}{q_2u}. \quad (\text{A.4})$$

(a) For g_1 we have

$$q_2 = -\frac{1}{s}; \quad q_1 = -\frac{1}{s(1+s)}; \quad q_0 = \frac{1}{1+s}; \quad (\text{A.5})$$

so

$$Q = -\frac{1}{s(1+s)}; \quad P = -\frac{2+s}{2(1+s)}; \quad (\text{A.6})$$

and

$$\begin{aligned} u'' + \frac{2+s}{s(1+s)}u' - \frac{1}{s(1+s)}u &= 0, \\ s(1+s)u'' + (2+s)u' - u &= 0. \end{aligned} \quad (\text{A.7})$$

We recognize this as the hypergeometric equation for a function $w(z)$ [5],

$$z(1-z)\frac{d^2w}{dz^2} + [c - (a+b+1)z]\frac{dw}{dz} - abw = 0, \quad (\text{A.8})$$

whose solution is

$$F(a, b; c; z) = 1 + \frac{ab}{c}z + \frac{a(a+1)b(b+1)}{c(c+1)}\frac{z^2}{2!} + \dots \quad (\text{A.9})$$

In particular, we have

$$u_1 = F(1, -1; 2; -s) = 1 + \frac{1}{2}s \quad (\text{A.10})$$

and finally, from Eq.(A.4), we have

$$g_1 = \frac{s}{2+s}. \quad (\text{A.11})$$

(b) For g_2 we have

$$q_2 = -\frac{1}{s}; \quad q_1 = -\frac{2s+1}{s(1+s)}; \quad q_0 = -\frac{1}{1+s}, \quad (\text{A.12})$$

so

$$Q = \frac{1}{s(1+s)}; \quad P = -\frac{2+3s}{2(1+s)}; \quad (\text{A.13})$$

and

$$\begin{aligned} u'' + \frac{2+3s}{s(1+s)}u' + \frac{1}{s(1+s)}u &= 0, \\ s(1+s)u'' + (2+3s)u' + u &= 0. \end{aligned} \quad (\text{A.14})$$

with the solution

$$u_2 = F(1, 1; 2; -s) = -\frac{1}{s} \ln(1+s), \quad (\text{A.15})$$

from which we find, from Eq.(A.4)

$$g_2 = -1 + \frac{s}{(1+s)\ln(1+s)}. \quad (\text{A.16})$$

This can be confirmed by direct substitution in the original differential equation.

B. Example of spherical geometry

The volume consists of the region between two concentric conducting spherical shells of radii a , b ($a < b$). Voltages are applied to the shells such that a radial electric field $\mathbf{E} = A\mathbf{r}/r^3$ is produced. It is assumed that the force on the ions is outward, unless stated to the contrary.

Consider the starting point to be at $\mathbf{R} = (R, 0, 0)$ at $t = 0$, with $a < R < b$. The drifting center moves outward along the $x = x_1$ axis. The classical equation of motion, $\mathbf{v} = \mu \mathbf{E}$, reduces to $dx/dt = \mu A/x^2$, with solution,

$$r_0(t) = (R^3 + 3\mu A t)^{1/3} \quad (\text{B.1})$$

We define

$$T = \frac{R^3}{3\mu A} \quad (\text{B.2})$$

Then we have

$$r_0 = R\left(1 + \frac{t}{T}\right)^{1/3} \quad (\text{B.3})$$

The needed partial derivatives of $2\mu \mathbf{E}$ near the x -axis are

$$\begin{aligned} 2\mu \partial_1 E_1 &= -\frac{2}{3T} \cdot \frac{1}{(1+t/T)} \\ 2\mu \partial_2 E_2 &= +\frac{1}{3T} \cdot \frac{1}{(1+t/T)} \end{aligned}$$

with the derivative in the x_3 direction the same as for x_2 . The equations for g_j , $j = 1, 2$ take the form

$$\frac{dg_j}{dt} + \frac{g_j(1+g_j)}{s} + \alpha \frac{(1+g_j)}{(1+s)} = 0 \quad (\text{B.4})$$

where, as before, $s = t/T$. The parameter α is $\alpha = -2/3, +1/3$ for the longitudinal ($j = 1$) and transverse ($j = 2$) directions, respectively.

With the introduction of $g_j = s \cdot w(s)$, the equation for g_j becomes a Riccati equation for $w(s)$:

$$\frac{dw}{ds} + w^2 + w\left(\frac{2}{s} + \frac{\alpha}{1+s}\right) + \frac{\alpha}{s(1+s)} = 0. \quad (\text{B.5})$$

With $z = -s$ and $w = \frac{1}{y} \frac{dy}{ds}$, this becomes the hypergeometric equation, Eq.(A.8), for $F(a, b; c; z)$ with $c = 2$, $a + b = 1 + \alpha$, $ab = \alpha$. Thus for $j = 1$, $a = 1$, $b = -2/3$, $c = 2$ and for $j = 2$, $a = 1$, $b = 1/3$, $c = 2$. With some algebra and use of relations of the hypergeometric functions, we find the solutions for the longitudinal and transverse functions, g_1 and g_2 :

$$g_1 = \frac{1}{3} \cdot \frac{s}{(1+s)} \cdot \frac{F(1, 1/3; 3; s/(1+s))}{F(1; -2/3; 2; s/(1+s))} \quad (\text{B.6})$$

$$g_2 = \frac{-1}{6} \cdot \frac{s}{(1+s)} \cdot \frac{F(1, 4/3; 3; s/(1+s))}{F(1; 1/3; 2; s/(1+s))} \quad (\text{B.7})$$

The classical center is given by $r_0(t) = R(1+s)^{1/3}$. For example, for $r_0(t)/R = 2$, we have $s = 7$.

Figures 7 and 8 show the behavior of $g_1(s)$ and $-g_2(s)$ versus s and the corresponding widths, $\sigma_{\parallel}/\sigma_0$ and σ_{\perp}/σ_0 , qualitatively similar to the cylindrical situation, but with differences stemming from the more rapid fall-off of the electric field with distance.

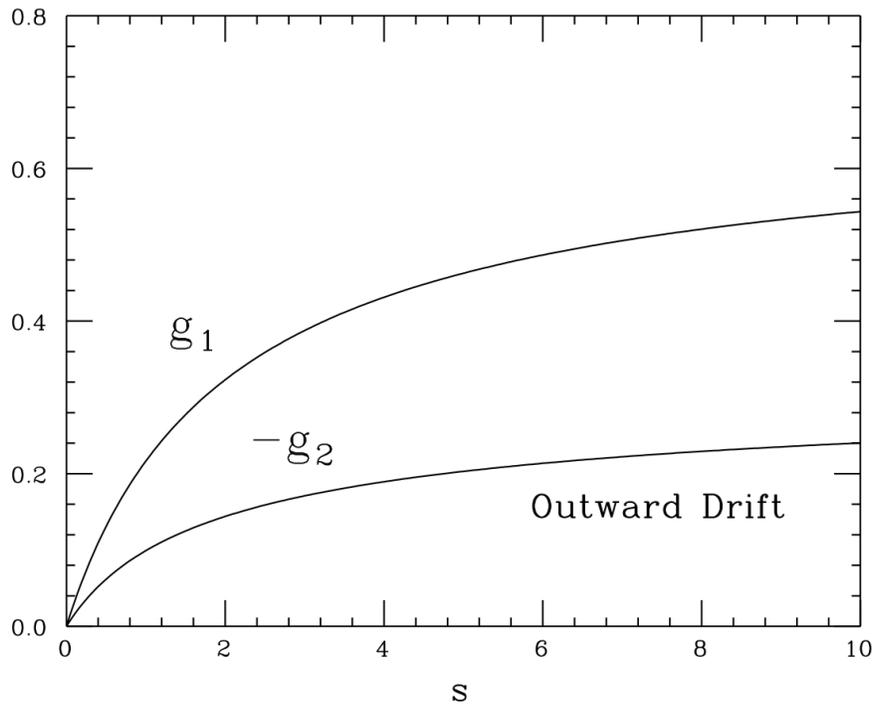


Figure 7. The functions appearing in the Gaussian description of ψ for spherical geometry and outward drift as functions of the dimensionless time variable $s = t/T$, where T is now given by Eq.(B.2).

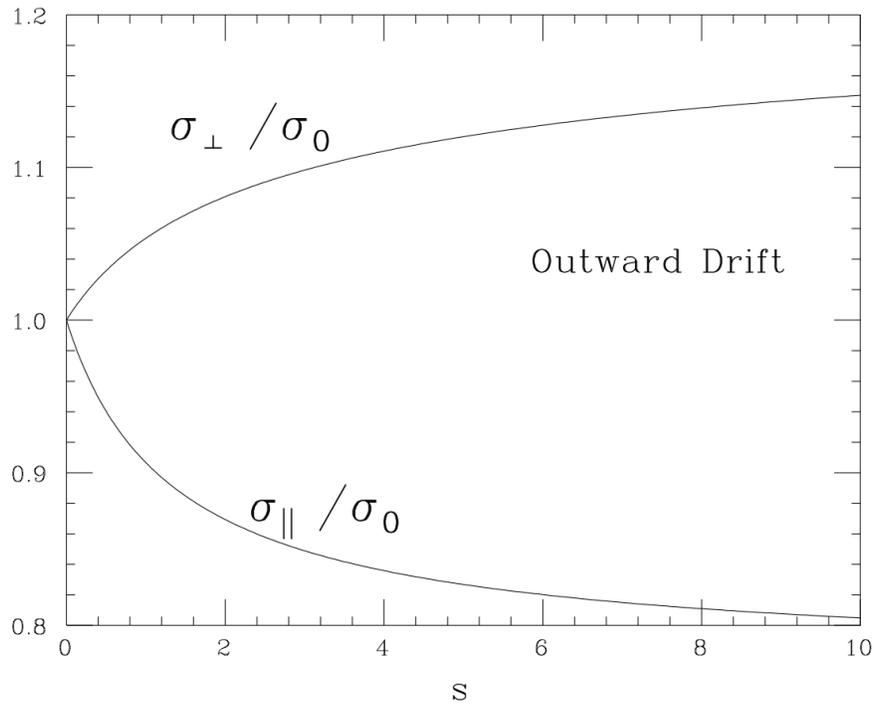


Figure 8. The effective dimensionless widths in the directions parallel and perpendicular to the radial direction for spherical geometry and outward drift as functions of the dimensionless time variable $s = t/T$, where T is now given by Eq.(B.2).

References

- [1] L. G. H. Huxley and R. W. Crompton, *The Diffusion and Drift of Electrons in Gases*, Wiley, New York, 1974.
- [2] E. A. Mason and E. W. McDaniel, *Transport Properties of Ions in Gases*, Wiley, New York, 1988.
- [3] Td stands for Townsend. $1 \text{ Td} = 10^{-17} \text{V} - \text{cm}^2$. E is the electric field in volts/cm; N is the number of fluid atoms per cubic centimeter.
- [4] H. R. Skullerud, *Longitudinal diffusion of electrons in electrostatic fields in gases*, J. Phys. B **2** (1969) 696-705 .
- [5] M. Abramowitz and I.A. Stegun, eds., *Handbook of Mathematical Functions*, National Bureau of Standards, Washington, 1964, p. 562, Eq. 15.5.1.
- [6] G. A. Korn and T. M. Korn, *Mathematical Handbook for Scientists and Engineers*, 2nd. ed, McGraw-Hill, N. Y., 1968, pp. 250-1.
- [7] http://en.wikipedia.org/wiki/Riccati_equation.

DISCLAIMER This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor The Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or The Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or The Regents of the University of California.